Online exam — Functional Analysis (WBMA033-05)

Tuesday 30 March 2021, 15.00h–18.00h CEST (plus 30 minutes for uploading)

University of Groningen

Instructions

1. Only references to the lecture notes and slides are allowed. References to other sources are *not* allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the exam grade is $G = 1 + p/10$.
4. Write both your name and student number on the answer sheets!
5. This exam comes in two versions. Both versions consist of five problems of equal difficulty.

   Make version 1 if your student number is odd.

   Make version 2 if your student number is even.

   For example, if your student number is 1277456, which is even, then you have to make version 2.
6. Please submit your work as a single PDF file.

Version 1 (for odd student numbers)

Problem 1 (5 + 10 + 5 = 20 points)

Equip the linear space $\mathcal{C}([0,1],\mathbb{K})$ with the following norms:

$$\|f\|_1 = \int_0^1 |f(x)|\,dx \quad \text{and} \quad \|f\|_3 = \left(\int_0^1 |f(x)|^3\,dx\right)^{1/3}.$$

(a) Show that $\|f\|_1 \le \|f\|_3$ for all $f \in \mathcal{C}([0,1],\mathbb{K})$.

(b) Consider the sequence (f_n) given by

$$f_n(x) = \begin{cases} n^{1/3} & \text{if } 0 \le x < 1/n, \\ x^{-1/3} & \text{if } 1/n \le x \le 1. \end{cases}$$

Compute $\|f_n\|_1$ and $\|f_n\|_3$ for all $n \in \mathbb{N}$.

(c) Are the norms $\|\cdot\|_1$ and $\|\cdot\|_3$ equivalent?

Problem 2 (5 + 3 + 7 + 5 = 20 points)

Consider the following linear operator:

$$T : \mathcal{C}([0,1],\mathbb{K}) \to \mathcal{C}([0,1],\mathbb{K}), \quad Tf(x) = f(x^2).$$

On the space $\mathcal{C}([0,1],\mathbb{K})$ we take the sup-norm $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$.

(a) Compute the operator norm of T.

(b) Show that $\lambda = 1$ is an eigenvalue of T.

(c) Is T invertible?

(d) Is T compact?

Problem 3 (12 points)

Let X and Y be Banach spaces, and let $T \in L(X,Y)$. Assume that

$$f \circ T \in X' \quad \text{for all} \quad f \in Y'.$$

(Clarification: $(f \circ T)(x) = f(Tx)$ for all $x \in X$.)

Use the Uniform Boundedness Principle to prove that T is bounded.

See next page for problems 4 and 5...

Problem 4 (8 + 6 + 6 + 8 = 28 points)

Let X be a Hilbert space over $\mathbb{C}$, and assume that $T \in B(X)$.

(a) Show that there exist selfadjoint operators $U, V \in B(X)$ such that $T = U + iV$.

(b) Show that T is normal if and only if $UV = VU$.

For parts (c) and (d) assume that T is normal.

(c) Show that $\|Tx\|^2 = \|Ux\|^2 + \|Vx\|^2$ for all $x \in X$.

(d) Show that if $0 \in \rho(U) \cup \rho(V)$, then $0 \in \rho(T)$.

Problem 5 (10 points)

Equip the linear space $X = \mathcal{C}([-1, 1], \mathbb{C})$ with the following norm:

$$\|f\| = \int_{-1}^{1} |f(x)|\, dx, \qquad f \in X.$$

Let $g(x) = e^{3ix}$. Prove that there exists a functional $\varphi \in X'$ such that

$$\varphi(g) = 4 - 2i \qquad \text{and} \qquad \|\varphi\| = \sqrt{5}.$$

End of test ("version 1", 90 points)

Version 2 (for even student numbers)

Problem 1 (5 + 10 + 5 = 20 points)

Equip the linear space $\mathcal{C}([0,1],\mathbb{K})$ with the following norms:

$$\|f\|_1 = \int_0^1 |f(x)|\,dx \quad \text{and} \quad \|f\|_5 = \left(\int_0^1 |f(x)|^5\,dx\right)^{1/5}.$$

(a) Show that $\|f\|_1 \leq \|f\|_5$ for all $f \in \mathcal{C}([0,1],\mathbb{K})$.

(b) Consider the sequence (f_n) given by

$$f_n(x) = \begin{cases} n^{1/5} & \text{if } 0 \leq x < 1/n, \\ x^{-1/5} & \text{if } 1/n \leq x \leq 1. \end{cases}$$

Compute $\|f_n\|_1$ and $\|f_n\|_5$ for all $n \in \mathbb{N}$.

(c) Are the norms $\|\cdot\|_1$ and $\|\cdot\|_5$ equivalent?

Problem 2 (5 + 3 + 7 + 5 = 20 points)

Consider the following linear operator:

$$T : \mathcal{C}([0,1],\mathbb{K}) \to \mathcal{C}([0,1],\mathbb{K}), \quad Tf(x) = f(\sqrt{x}).$$

On the space $\mathcal{C}([0,1],\mathbb{K})$ we take the sup-norm $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$.

(a) Compute the operator norm of T.

(b) Show that $\lambda = 1$ is an eigenvalue of T.

(c) Is T invertible?

(d) Is T compact?

Problem 3 (12 points)

Let X and Y be Banach spaces, and let $T \in L(X,Y)$. Assume that

$$f \circ T \in X' \quad \text{for all} \quad f \in Y'.$$

(Clarification: $(f \circ T)(x) = f(Tx)$ for all $x \in X$.)

Use the Uniform Boundedness Principle to prove that T is bounded.

See next page for problems 4 and 5...

Problem 4 (8 + 6 + 6 + 8 = 28 points)

Let X be a Hilbert space over $\mathbb{C}$, and assume that $T \in B(X)$.

(a) Show that there exist selfadjoint operators $U, V \in B(X)$ such that $T = U + iV$.

(b) Show that T is normal if and only if $UV = VU$.

For parts (c) and (d) assume that T is normal.

(c) Show that $\|Tx\|^2 = \|Ux\|^2 + \|Vx\|^2$ for all $x \in X$.

(d) Show that if $0 \in \sigma(T)$, then $0 \in \sigma(U) \cap \sigma(V)$.

Problem 5 (10 points)

Equip the linear space $X = \mathcal{C}([0,2], \mathbb{C})$ with the following norm:

$$\|f\| = \int_0^2 |f(x)|\, dx, \qquad f \in X.$$

Let $g(x) = e^{-5ix}$. Prove that there exists a functional $\varphi \in X'$ such that

$$\varphi(g) = -6 + 2i \qquad \text{and} \qquad \|\varphi\| = \sqrt{10}.$$

End of test ("version 2", 90 points)

Solution of problem 1 (5 + 10 + 5 = 20 points)

Equip the linear space $\mathcal{C}([0,1],\mathbb{K})$ with the following norms:

$$\|f\|_1 = \int_0^1 |f(x)|\,dx \quad \text{and} \quad \|f\|_p = \left(\int_0^1 |f(x)|^p\,dx\right)^{1/p},$$

where $p > 1$. The sequence (f_n) is given by

$$f_n(x) = \begin{cases} n^{1/p} & \text{if } 0 \le x < 1/n, \\ x^{-1/p} & \text{if } 1/n \le x \le 1. \end{cases}$$

So in version 1 and 2 we have $p = 3$ and $p = 5$, respectively.

(a) With $1/p + 1/q = 1$ we have Hölder's inequality:

$$\int_0^1 |f(x)g(x)|\,dx \le \left(\int_0^1 |f(x)|^p\,dx\right)^{1/p}\left(\int_0^1 |g(x)|^q\,dx\right)^{1/q}.$$

(3 points)

In particular, with $g(x) = 1$ for all $x \in [0,1]$ we obtain

$$\int_0^1 |f(x)|\,dx \le \left(\int_0^1 |f(x)|^p\,dx\right)^{1/p},$$

which proves the desired inequality.
(2 points)

(b) We have that

$$\begin{aligned} \|f_n\|_1 &= \int_0^{1/n} n^{1/p}\,dx + \int_{1/n}^1 x^{-1/p}\,dx \\ &= n^{-\frac{p-1}{p}} + \left[\tfrac{p}{p-1} x^{\frac{p-1}{p}}\right]_{1/n}^1 \\ &= \tfrac{p}{p-1} - \tfrac{1}{p-1}\cdot n^{-\frac{p-1}{p}}. \end{aligned}$$

Version 1. For $p = 3$ we have $\|f_n\|_1 = \frac{3}{2} - \frac{1}{2}n^{-\frac{2}{3}}$.

Version 2. For $p = 5$ we have $\|f_n\|_1 = \frac{5}{4} - \frac{1}{4}n^{-\frac{4}{5}}$.

(5 points)

We have that

$$\|f_n\|_p^p = \int_0^{1/n} n\,dx + \int_{1/n}^1 x^{-1}\,dx = 1 + \big[\log(x)\big]_{1/n}^1 = 1 + \log(n),$$

which gives $\|f_n\|_p = (1 + \log(n))^{1/p}$.

Version 1. For $p = 3$ we have $\|f_n\|_3 = (1 + \log(n))^{1/3}$.

Version 2. For $p = 5$ we have $\|f_n\|_5 = (1 + \log(n))^{1/5}$.

(5 points)

(c) From part (a) we know that $\|f\|_1 \leq \|f\|_p$ for all $f \in \mathcal{C}([0,1],\mathbb{K})$. Therefore, the norms $\|\cdot\|_1$ and $\|\cdot\|_3$ are equivalent if and only if there exists a constant $c > 0$ such that

$$\|f\|_p \leq c\|f\|_1 \quad \text{for all} \quad f \in \mathcal{C}([0,1],\mathbb{K}).$$

In particular, for the sequence of part (b) we must have

$$\|f_n\|_p \leq c\|f_n\|_1 \quad \text{for all} \quad n \in \mathbb{N}.$$

But this is a contradiction since the left hand side is unbounded, whereas the right hand side is bounded. Therefore, the two norms are not equivalent.
(5 points)

Solution of problem 2 (5 + 3 + 7 + 5 = 20 points)

(a) Since the function $x \mapsto x^2$ maps the interval $[0,1]$ bijectively onto itself we have

$$\|Tf\|_\infty = \sup_{x\in[0,1]} |Tf(x)| = \sup_{x\in[0,1]} |f(x^2)| = \sup_{x\in[0,1]} |f(x)| = \|f\|_\infty.$$

(3 points)

Therefore, the operator norm of T is given by

$$\|T\| = \sup_{f\neq 0} \frac{\|Tf\|_\infty}{\|f\|_\infty} = 1.$$

(2 points)

(b) The equality $f(x) = f(x^2)$ holds for all constant functions. Therefore, any nonzero constant function f is an eigenvector for the eigenvalue $\lambda = 1$.
(3 points)

(c) Consider the operator

$$S : \mathcal{C}([0,1],\mathbb{K}) \to \mathcal{C}([0,1],\mathbb{K}), \quad Sf(x) = f(\sqrt{x}).$$

We have

$$STf(x) = f(\sqrt{x^2}) = f(x) \quad \text{and} \quad TSf(x) = f(\sqrt{x}^2) = f(x),$$

which means that $ST = TS = I$.
(4 points)

By a similar argument as in part (a) it follows that S is bounded. Therefore, the operator T is invertible.
(3 points)

(d) *Method 1.* The space $\mathcal{C}([0,1],\mathbb{K})$ is infinite-dimensional. If T were compact, then we would have $0 \in \sigma(T)$. However, in part (c) we have established that T is invertible, which means that $0 \in \rho(T)$. Therefore, T is not compact.
(5 points)

Method 2. If T is compact, then so is $I = TT^{-1}$. But then the closed unit ball is compact. However, this is not possible because the space $\mathcal{C}([0,1],\mathbb{K})$ is infinite-dimensional. Therefore, T is not compact.
(5 points)

Solution of problem 3 (12 points)

Let $x \in X$ be arbitrary. By a consequence of the Hahn Banach theorem we have

$$\begin{aligned}\sup\{|(f \circ T)(x)| \,:\, f \in Y',\, \|f\| = 1\} &= \sup\{|f(Tx)| \,:\, f \in Y',\, \|f\| = 1\} \\ &= \|Tx\| < \infty.\end{aligned}$$

(3 points)

The Uniform Boundedness Principle implies that

$$c := \sup\{\|f \circ T\| \,:\, f \in Y',\, \|f\| = 1\} < \infty.$$

(3 points)

In particular, if $f \in Y'$ has norm $\|f\| = 1$, then

$$|(f \circ T)(x)| \leq \|f \circ T\| \, \|x\| \leq c\|x\|.$$

(3 points)

Taking the supremum over all such elements f gives

$$\|Tx\| = \sup\{|(f \circ T)(x)| \,:\, f \in Y',\, \|f\| = 1\} \leq c\|x\|.$$

Since $x \in X$ is arbitrary, it follows that T is bounded.
(3 points)

Solution of problem 4 (8 + 6 + 6 + 8 = 28 points)

(a) Define the operators

$$U = \frac{1}{2}(T + T^*) \quad \text{and} \quad V = \frac{1}{2i}(T - T^*).$$

Clearly, $U, V \in B(X)$ since they are linear combinations of the bounded operators T and T^*.
(2 points)

The operator U is selfadjoint since

$$U^* = \frac{1}{2}(T^* + T) = \frac{1}{2}(T + T^*) = U$$

(2 points)

The operator V is selfadjoint since

$$V^* = -\frac{1}{2i}(T^* - T) = \frac{1}{2i}(T - T^*) = V.$$

(2 points)

Finally, we have that

$$U + iV = \frac{1}{2}(T + T^*) + \frac{1}{2}(T - T^*) = T.$$

(2 points)

(b) The adjoint of $T = U + iV$ is given by $T^* = U^* - iV^* = U - iV$. Computing their products gives

$$\begin{aligned} T^*T &= (U - iV)(U + iV) = U^2 + V^2 + i(UV - VU), \\ TT^* &= (U + iV)(U - iV) = U^2 + V^2 + i(VU - UV). \end{aligned}$$

(3 points)

By definition, T is normal when $T^*T = TT^*$. This holds if and only if

$$UV - VU = VU - UV,$$

or, equivalently, $UV = VU$.
(3 points)

(c) Since T is normal, we have that $UV = VU$. This gives

$$\|Tx\|^2 = (Tx, Tx) = (T^*Tx, x) = ((U^2 + V^2)x, x) = (U^2x, x) + (V^2x, x).$$

(3 points)

Since U and V are selfadjoint, we have

$$(U^2x, x) + (V^2x, x) = (Ux, Ux) + (Vx, Vx) = \|Ux\|^2 + \|Vx\|^2.$$

(3 points)

(d) *Version 1.* Assume that $0 \in \rho(U)$. (In case $0 \in \rho(V)$ we can argue similarly.)

Since U is selfadjoint, and thus normal, there exists a constant $c > 0$ such that $\|Ux\| \geq c\|x\|$ for all $x \in X$. (This follows from the characterization of the resolvent set for a normal operator.)
(4 points)

By part (c) it follows for all $x \in X$ that

$$\|Tx\|^2 = \|Ux\|^2 + \|Vx\|^2 \geq c^2\|x\|^2,$$

which implies that $\|Tx\| \geq c\|x\|$. Since T is normal, we conclude that $0 \in \rho(T)$.
(4 points)

Version 2. Assume that $0 \in \sigma(T)$. Since T is normal, it follows that $\lambda = 0$ is an approximate eigenvalue of T. Therefore, there exists a sequence (x_n) in X such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\|Tx_n\| \to 0$.
(4 points)

By part (c) it follows that

$$\|Ux_n\|^2 \leq \|Ux_n\|^2 + \|Vx_n\|^2 = \|Tx_n\|^2 \to 0,$$

which means that $\lambda = 0$ is also an approximate eigenvalue of U. Therefore, $0 \in \sigma(U)$. By the same reasoning, we have that $0 \in \sigma(V)$.
(4 points)

Solution of problem 5 (10 points)

Version 1. Define the map

$$\varphi : \operatorname{span}\{g\} \to \mathbb{C}, \quad \varphi(\lambda g) = \lambda(4-2i).$$

With $\lambda = 1$ we have that $\varphi(g) = 4 - 2i$.
(2 points)

Since $\|g\| = 2$ we have that

$$\|\varphi\| = \sup_{\lambda \neq 0} \frac{|\varphi(\lambda g)|}{\|\lambda g\|} = \sup_{\lambda \neq 0} \frac{|\lambda|\sqrt{20}}{2|\lambda|} = \sqrt{5}.$$

(5 points)

Now apply the Hahn-Banach theorem to extend φ to the entire space X while preserving the norm.
(3 points)

Version 2. Define the map

$$\varphi : \operatorname{span}\{g\} \to \mathbb{C}, \quad \varphi(\lambda g) = \lambda(-6+2i).$$

With $\lambda = 1$ we have that $\varphi(g) = -6 + 2i$.
(2 points)

Since $\|g\| = 2$ we have that

$$\|\varphi\| = \sup_{\lambda \neq 0} \frac{|\varphi(\lambda g)|}{\|\lambda g\|} = \sup_{\lambda \neq 0} \frac{|\lambda|\sqrt{40}}{2|\lambda|} = \sqrt{10}.$$

(5 points)

Now apply the Hahn-Banach theorem to extend φ to the entire space X while preserving the norm.
(3 points)